Constrained Biogeography-Based Optimization for Invariant Set Computation

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***Abstract*— We discuss the application of biogeography-based optimization (BBO) to invariant set approximation. BBO is a recently developed evolutionary algorithm (EA) that is motivated by biogeography, which is the study and science of the geographical migration of biological species. Invariant sets are sets in the state space of a dynamic system such that if the state begins in the set, then it remains in the set for all time. Invariant sets have applications in many constrained control problems, and their computation amounts to a constrained optimization problem. We therefore frame the invariant set computation problem as a constrained optimization problem, and we use a constrained BBO algorithm to solve it. We study three specific invariant set problems: the approximation of the maximum invariant ellipsoid, the approximation of the maximum invariant semi-ellipsoid, and the approximation of the maximum invariant cylinder, which has application to sliding mode control. We find that BBO outperforms linear matrix inequality (LMI) algorithms for the first and third of these problems. For the second problem, LMI performs better than BBO, but BBO only requires 65% of the computational effort.**

I. Introduction

Biogeography-based optimization (BBO) is an evolutionary optimization algorithm that is based on the science and study of biogeography. According to the science of island biogeography, species migrate from one island to another. This process of migration results in more habitable islands, because species diversity generally improves the habitability of islands [1], [2]. The environmental factors of an island, like the amount of rainfall and the temperature range, are called suitability index variables (SIVs), and determine the habitability of an island. A habitat that is more suitable for species has a higher habitat suitability index (HSI) and contains more species. Habitats with a low HSI support fewer species. Migration depends on the habitat's HSI.

Habitats with a high HSI have high emigration rates and low immigration rates; that is, they "send out" many species to other islands because of the accumulation of probabilistic effects on the dispersal patterns of their high population, but they cannot accept many new species because they are already saturated with nearly as many species as they are able to support. Habitats with a low HSI have a low emigration rate and a high immigration rate; that is, they do not send out many species to other islands because of their small population, but they tend to accept incoming species because they have a lot of room to support additional life forms.

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BBO is the application of biogeography theory to the field of engineering optimization [3], [4]. In BBO, an island represents a candidate solution (also called an individual) to an optimization problem, HSI represents the fitness of the candidate solution, and SIVs represent the independent variables (also called decision variables, or features) of each candidate solution. If the optimization problem is a maximization problem, then the objective is to maximize fitness. If the optimization problem is a minimization problem, then the objective is to minimize cost. These problems are easily interchangeable, because minimizing a quantity is equivalent to maximizing its negative, and maximizing a quantity is equivalent to minimizing its negative. In BBO, an individual with high fitness (or low cost) has good results on the optimization problem, and so it has a high emigration rate and a low immigration rate. That is, it is likely to share its solution features with other individuals, but unlikely to modify its solution features on the basis of other individuals. A BBO individual with low fitness (or high cost) has poor results on the optimization problem, and so it has a high immigration rate and a low emigration rate. That is, it is likely to replace its solution features with those of other individuals, but unlikely to share its solution features with other individuals.

A. Unconstrained BBO

Unconstrained BBO is equivalent to the original BBO algorithm. There are three characteristics of BBO: migration, mutation, and elitism. As described above, migration allows an individual to immigrate or emigrate its solution features during each generation (or iteration) of the optimization process. As mentioned above, an individual's migration rate depends on its fitness. Although nonlinear curves can be used to map fitness to migration rate [5], in this paper we use linear functions to determine migration rates. If we use $f(x)$ to denote the fitness of candidate solution x, then the emigration rate μ and the immigration rate λ of a given candidate solution y are determined as follows:

$$\begin{aligned} \mu(y) &= \frac{f(y) - \min_x f(x)}{\max_x f(x) - \min_x f(x)} \\ \lambda(y) &= 1 - \mu(y) \end{aligned} \tag{1}$$

where the minimization and maximation are taken over the entire population of candidate solutions. This gives $\mu \in [0, 1]$, where $\mu = 0$ for the least fit individual, and $\mu = 1$ for the most fit individual. Conversely, $\lambda = 1$ for the least fit individual, and $\lambda = 0$ for the most fit individual. For each solution feature in each candidate solution x, we use $\lambda(x)$ to decide whether or not to immigrate:

```
If rand(0,1) < λ(x) then
    Immigrate to x(v)
else
    Do not immigrate to x(v)
End if
```

In the above, rand(0,1) is a random number that is selected from a uniform distribution on $[0, 1]$, and $x(v)$ is a solution feature of x. Immigration to $x(v)$ means that we replace it with a solution feature that is probabilistically selected from the rest of the population based on the emigration rates:

$$\text{Prob(emigration from } y) = \frac{\mu(y)}{\sum_x \mu(x)} \text{ for all } y. \quad (2)$$

As in other EAs, BBO also includes mutation, which randomly modifies solution features. There are many possible implementations of mutation in EAs. This probabilistic operation is relatively rare, but is required to insert new information into the popuIuation to increase the exploratory capability of the optimization algorithm. The following algorithm is a description of one generation of BBO.

```
For each candidate solution x_i
  Calculate immigration and emigration rate λ_i and μ_i
Next candidate solution: i ← i + 1
For each candidate solution x_i
  For each solution feature v in x_i
    Use λ_i to decide whether to immigrate to x_i
    If immigrating to x_i then
      Use {μ_j} to select emigrating solution x_k
      x_k emigrates data to x_i : x_i(v) ← x_k(v)
    End immigration
  Next solution feature v
  Probabilistically mutate x_i
Next candidate solution: i ← i + 1
```

Although not shown in the above algorithm, like other EAs, BBO typically includes elitism, which saves the best individual at each generation. This ensures that the best fitness of the population is monotonically nondecreasing from one generation to the next. There are several ways to implement elitism, but generally we implement it by replacing the worst individuals at each generation with the best individuals from the previous generation. The number of elites is usually about two.

The above algorithm is executed for many generations. As in other EAs, the termination critera of the BBO algorithm can include several conditions, depending on the user's preference. For example, we could terminate after a fixed number of generations, or after the best candidate solution stops improving, or after a solution of desired quality is found.

B. Constrained BBO

All real-world optimization problems are constrained, at least implicitly, if not explicitly. This section gives a brief overview of constraint handling in EAs, which are a superset of BBO. A constrained optimization problem can be written as

$$\min_x f(x) \quad \text{such that} \quad \begin{aligned} g_i(x) &\leq 0 \text{ for } i \in [1, m] \\ h_j(x) &= 0 \text{ for } j \in [1, p] \end{aligned} \quad (3)$$

This problem includes $(p + m)$ constraints, m of which are inequality constraints, and p of which are equality constraints. The set of x that satisfies all $(p+m)$ constraints is called the feasible set, and the set of x that violates one or more constraints is called the infeasible set.

In this paper, we focus on penalty function approaches in constrained BBO. We can design a penalty function method in at least two different ways. First, we could penalize feasible individuals as they move closer to the constraint boundary; these are called interior point methods, or barrier methods. This approach does not allow any infeasible individuals in the population. Interior point methods are not used very often in constrained EAs. This is because for many constrained problems, finding candidate solutions that satisfy all of the constraints is itself a challenging problem. Also, infeasible solutions may include information that is valuable in the search for a constrained optimum.

This leads us to the second penalty function approach, in which we allow infeasible individuals in the population, but penalize them in terms of cost, or in terms of selection for contributing to the next generation. This approach generally does not penalize feasible individuals, no matter how close they are to the constraint boundary. Such approaches are called exterior methods.

II. Overview of Invariant Set Concepts

The constrained optimization problem that we solve in this paper is the computation of invariant sets. Invariant sets are mathematical constructs that arise naturally in many systems and control settings. An example is given by the famous invariance principle due to LaSalle [6]. Invariant sets are also introduced as design and analysis tools in the area of control systems with constraints. The central definition is that of a *positively invariant* (PI) set. Given a dynamical system evolving in a state space X, a set $\mathcal{S} \subset X$ is PI if all initial conditions $x(0) \in \mathcal{S}$ give rise to trajectories $x(t)$ satisfying $x(t) \in \mathcal{S}$ for all $t \geq 0$.

For systems given by the differential equation $\dot{x} = f(x, t)$, a theorem by Nagumo [7] provides a necessary and sufficient condition for a set to be PI. The condition is based in the concept of tangent cone and applies to sets of rather general shapes. In this paper, we focus in half-spaces and ellipsoids as primary PI sets and state the corresponding invariance conditions. A related concept is that of a *robustly and positively invariant* (RPI) set. In this case, the system is given by the differential equation $\dot{x} = f(x, \zeta(t), t)$, where $\zeta(t)$ is an uncertain input with values in a set $\mathcal{Z}$. A set is RPI if the PI property holds for all systems $\dot{x} = f_z(x, z, t)$, where $z \in \mathcal{Z}$ is constant.

When $\mathcal{S}$ is an interval of the form $[a, b]$, the PI condition for the autonomous system $\dot{x} = f(x)$ has a rather intuitive form: $f(a) \geq 0$ and $f(b) \leq 0$. That is, if $x(0) \in [a, b]$, $x(t)$ will remain in the interval for $t > 0$ if $\dot{x}$ is non-negative at the boundary $x = a$ and non-positive at the boundary $x = b$.

Invariance of a Half-Space

Consider an autonomous system of the form $\dot{x} = f(x)$ and the set defined by all points x that satisfy the inequality $G_i x \leq 1$, where G_i is a row vector. Nagumo's theorem reduces to the following condition for invariance:

$$G_i f(x) \leq 0 \text{ along } G_i x = 1 \tag{4}$$

For linear state-space systems (A, B) under state feedback $u = -Kx$, we have $f(x) = (A - BK)x$, and the PI condition reduces to

$$G_i(A - BK)x \leq 0 \text{ along } G_i x = 1 \tag{5}$$

A. Invariance of an Ellipsoid

An n-dimensional ellipsoid $\mathcal{E}$ is described by the inequality $x^T P x \leq 1$, where P is a symmetric, positive-definite matrix. Consider first an autonomous, linear state-space system described by $\dot{x} = Ax$ and an ellipsoidal set described by $\mathcal{E} = \{x \ : \ x^T P x \leq 1\}$. For $\mathcal{E}$ to be PI, Nagumo's theorem requires that $\dot{x} = f(x) = Ax$ be directed toward the interior of $\mathcal{E}$, when x is taken at its boundary, defined by $h(x) = x^T P x = 1$. This can be ensured by the condition that the projection of $f(x)$ onto a vector normal to the boundary be negative. The invariance condition is expressed as $\nabla h(x).f(x) < 0$. For an ellipsoidal boundary we have $\nabla h(x) = 2x^T P$, thus the condition becomes $2x^T PAx < 0$. Symmetry of P can be used to reduce the invariance condition to the *Lyapunov inequality*

$$PA + A^T P < 0 \tag{6}$$

The above condition can be applied to evaluate the invariance of an ellipsoid relative to the closed loop system $\dot{x} = (A - BK)x$ resulting from using the state feedback control law $u = -Kx$:

$$P(A - BK) + (A - BK)^T P < 0 \tag{7}$$

All ellipsoids of the form $x^T P x = a$, with $a > 0$, are then PI relative to the closed-loop dynamics.

B. Constrained Control and Maximal Admissible Sets

Control design problems considering input, output and state constraints are commonplace. Constraints can be imposed on the range of these variables or on their rates. For example, gas turbine engines must be operated by means of feedback based on sensed mechanical speeds, using fuel flow rate as the control input [8], [9]. Input constraints reflect the maximum and minimum flow rates that can be delivered by the fuel pump. Also, fuel flow rate cannot rise or decrease faster than certain amounts, introducing input rate constraints. Since commonly-used engine models use mechanical speeds as states, the requirement that shaft speeds be kept under maximum limits corresponds to state magnitude constraints. Combinations of states and inputs (system outputs) are also subject to range limits, the compressor stall margin being an example. When linear state feedback is used, input constraints can be reduced to state constraints.

State constraints are usually formulated as a set of linear inequalities in the state variables: $\mathcal{G} = \cap \mathcal{G}_i$ for $i = 1, 2, \cdots, k$, where $\mathcal{G}_i = \{x \ : \ G_i x \leq 1\}$.

In general, Eq. 5 cannot be satisfied for all points belonging to the boundary defined by $G_i x = 1$. Geometrically, if G_i is not parallel to $G_i(A - BK)$, the boundary will be divided into three subsets: a subset where $G_i(A - BK)x = 0$, a subset where $G_i(A - BK) > 0$ and a subset where the condition is satisfied. When several constraints exist, methodologies have been developed to ensure that points belonging to the set where $G_i(A - BK)x > 0$ do not satisfy the other constraints. *Polyhedral invariant set theory* provides means to construct invariant sets using linear segments. Given a constraint set defined by design requirements, polyhedral invariant sets can be constructed with little conservatism. The interested reader is referred to Blanchini's survey for an introduction [10].

This paper focuses on invariant set constructions called *recoverable sets*, used in conjunction with feedback regulation in the presence of state and input constraints. Suppose a feedback law $u = k(x)$ has been specified for a sytem $\dot{x} = f(x, u)$ so that the origin is rendered asymptotically stable in a region $\mathcal{D}$ of the state space. Given a constraint set $\mathcal{G} \subset \mathcal{D}$ containing the origin in its interior and a maximum control magnitude U, a recoverable set $\mathcal{R} \subset \mathcal{G}$ contains initial conditions $x(0)$ such that $x(t) \in \mathcal{G}$ and $||k(x(t)|| \leq U$ for all $t \geq 0$. That, is trajectories starting in $\mathcal{R}$ proceed to the origin without leaving the constraint set $\mathcal{G}$ or exceeding the control bound. A *maximal recoverable set* is simply the set of *all* such $x(0)$. Note that a recoverable set does not need to be PI, however, the PI condition is usually imposed on $\mathcal{R}$ for tractability.

C. Sample Maximal Recoverable Set Constructions

Three invariant set constructions have been chosen in this paper to illustrate the application of BBO techniques. The first two constructions, namely ellipsodial and semi-ellipsoidal sets, apply to linear systems under linear state feedback, with state and input constraints. Ellipsoidal sets have a simple description, but are conservative, since their boundaries are poor approximations to the complex boundaries of true maximal recoverable sets. Semi-ellipsoidal sets were introduced by O'Dell [11] as a good tradeoff between conservative ellipsoidal sets and polyhedral sets, which offer low conservatism at the expense of high complexity. The third construction is the cylindrical invariant set introduced by Richter [12], applicable to linear systems with disturbance input, where sliding mode regulation is used. As elaborated below, the computation of maximal cylindrical recoverable sets represents a challenging numerical optimization problem.

The three set constructions are based on a state-space linear system of the form

$$\dot{x} = Ax + Bu + B\zeta(t) \tag{8}$$

where $x \in \mathbb{R}^n$ and u is a scalar control input. Ellipsoidal and semi-ellipsoidal constructions do not consider the disturbance input $\zeta(t)$.

Linear State Feedback: Ellipsoidal Sets: A simple approach to finding an operating set is to find the largest invariant ellipsoid contained in the constraint set. Assuming an input magnitude constraint of the form $|u| = |Kx| \leq U$, the optimization problem takes the form [11]:

$$\begin{aligned} \min_{K,\ P=P^T>0} \quad & \log\det P \qquad \text{subject to:} \\ P(A-BK)+(A-BK)^T P &< 0 \\ \Gamma_i P^{-1} \Gamma_i^T &\leq 1 \quad i=1,2,\cdots,k \\ K P^{-1} K^T &\leq U^2 \end{aligned} \tag{9}$$

Note that minimizing the determinant of P minimizes the product of its eigenvalues (all positive). Since the principal axes of an ellipsoid have lengths equal to the reciprocal of the eigenvalues, the volume of the ellipsoid is maximized. The logarithm function is used to improve convergence [13]. The first inequality constraint reflects the invariance condition of Eq. 7. The second and third constraints, fully derived in [13], incorporate the requirement that the ellipsoid be contained in the constraint set and account for the control bound.

Linear State Feedback: Semi-Ellipsoidal Sets: The requirement that the invariant ellipsoid be contained in the constraint set is rather conservative. A significant improvement is achieved by allowing $\mathcal{E}$ to exceed $\mathcal{G}$, provided the set $\mathcal{E} \cap \mathcal{G}$ retains the PI property. This condition implies that $\mathcal{E} \cap \mathcal{G}$ is recoverable. The boundary of this set is a mix of constraint boundaries and ellipsoidal boundaries, a shape better suited to approximate the true maximal recoverable set. As developed in [11], [13], the optimization problem takes the form:

$$\begin{aligned} \min_{K,\ P=P^T>0} \quad & \log\det P \qquad \text{subject to:} \\ P(A-BK)+(A-BK)^T P &< 0 \\ (\Gamma_i W_i)(W_i^T P W_i)^{-1}(\Gamma_i W_i)^T &\leq 1 \quad i=1,2,\cdots,k \\ K P^{-1} K^T &\leq \hat{u}^2 \end{aligned} \tag{10}$$

where W_i is a matrix whose columns are an orthonormal basis for $\text{null}(\Gamma_i(A-BK))$. In this formulation, the maximum control constraint is calculated using the boundary of $\mathcal{E}$ instead of that of $\mathcal{E} \cap \mathcal{G}$ to keep the problem tractable. As a result, the available bound U might be under-utilized. To address this problem, a fictitious control bound $\hat{u} \geq U$ is chosen iteratively, solving the above optimization problem to determine P and K. The maximum of $|Kx|$ over $\mathcal{E} \cap \mathcal{G}$ is then calculated using a separate program. If the result is less than U, optimization is repeated with a higher value of $\hat{u}$. O'Dell used a linear search formula to update $\hat{u}$ on the basis of previous values of $\hat{u}$ and the maximum of $|Kx|$.

Sliding Mode Control: Invariant Cylinders: Linear systems under sliding mode control (SMC) have naturally-fitting shapes for invariant sets [12]. RPI sets for n-th order systems are cylinders whose "cross-sections" are $(n-1)$-dimensional ellipsoids and whose "axis" is an arbitrary real interval containing zero. Mathematically, an invariant cylinder is defined by

$$\mathcal{H}_{\mathcal{E}}[a,b] = \left\{ [w_1^T | w_2]^T \in \mathbb{R}^n \ : \ w_1 \in \mathcal{E},\ w_2 \in [a,b] \right\} \tag{11}$$

where $\mathcal{E}$ is the ellipsoidal cross section:

$$\mathcal{E} = \left\{ w_1 \in \mathbb{R}^{n-1} \ : \ w_1^T P w_1 \leq 1 \right\} \tag{12}$$

and $[a,b]$ is an arbitrary real interval containing zero. Note that the interval may be extended to coincide with the real numbers, resulting in an infinite cylinder. The paper by Richter [12] showed that overall cylinder robust invariance is obtained by making $\mathcal{E}$ itself RPI. As described in [12], the dynamics of the sliding mode have a simple description when convenient coordinates are chosen. State variables w_1 evolve according to

$$\dot{w}_1 = A_w w_1 + B_1 \zeta'(t) \tag{13}$$

where A_w is Hurwitz and $\zeta'(t) = \eta + \bar{\zeta}$ is a generalized disturbance term. Parameters η and $\bar{\zeta}$ are respectively the *switching gain* used in sliding mode control laws and $\bar{\zeta}$ is a bound for the exogenous disturbance [12]. Matrices A_w and B_1 are calculated from the linear plant equations, SMC parameters and coordinate transformation matrices. RPI of the $(n-1)$-dimensional ellipsoid is equivalent to the existence of a scalar $\alpha > 0$ and a symmetric matrix $P > 0$ satisfying the following *quadratic boundedness criterion* [14]:

$$PA_w + A_w^T P + \alpha P + \alpha^{-1} \bar{\zeta}'^2 B_1 B_1^T P \leq 0 \tag{14}$$

Scalar α is upper-bounded by the α_0, the maximum decay rate associated with (A_w, B_1). If λ is the eigenvalue of A_w with the largest real part in absolute value, then $\alpha_0 = 2|\text{Re}(\lambda)|$. In [12], Eq. 14 and the $P > 0$ constraint are expressed in an equivalent Linear Matrix Inequality (LMI) form. An invariant cylinder can be generated by fixing α and solving an LMI feasibility problem.

Recoverable sets may be constructed in analogy to the semi-ellipsoidal set concept to account for state constraints. Conditions were derived in [12] so that the intersection between the infinite RPI cylinder and a convex set of state constraints retains the RPI property. These conditions are given by a set of inequalities that must be checked for each individual constraint. Due to space limitations, the set of inequalities from [12] is not reproduced here, but their evaluation can be regarded as a function $c(P, \alpha, \Gamma_i)$ returning 0 when the constraint passes all conditions and 1 when not. The complexity and lack of structure of c does not allow the determinination of explicit update laws for P and α so that a failing constraint becomes qualified. This feature makes BBO attractive as an optimization approach. The search for a maximal cylindrical recoverable set corresponds to the optimization problem:

$$\begin{aligned} \min_{\alpha,\ P=P^T>0} \quad & \log\det P \qquad \text{subject to:} \\ PA_w + A_w^T P + \alpha P + \alpha^{-1} \bar{\zeta}'^2 B_1 B_1^T P &\leq 0 \\ c(P, \alpha, \Gamma_i) &= 0 \quad i=1,2,\cdots,k \end{aligned} \tag{15}$$

III. Application of BBO to Invariant Set Approximation

We use an exterior penalty function method (as outlined in Section I-B) in conjunction with BBO (as outlined in Section I-A) to solve the invariant set optimization problems of Section II-C. We implemented constrained BBO by sorting all BBO individuals first according to their number of constraint violations $c(x)$, and secondarily according to their cost $f(x)$. We then rank the individuals. x is ranked better than y if

$$c(x) < c(y), \text{ or } c(x) = c(y) \text{ and } f(x) < f(y). \tag{16}$$

After we rank each BBO solution, the best individual has a rank of 1, and the worst individual has a rank of N, where N is the BBO population size. We then assign migration rates on the basis of normalized cost:

$$\begin{aligned} \mu(x) &= \frac{N - \text{rank}(x)}{N-1} \\ \lambda(x) &= 1 - \mu(x). \end{aligned} \tag{17}$$

This constraint-handling approach is similar to the niched-penalty approach for constrained optimization [15], [16], although that approach used tournament selection in a constrained genetic algorithm. It is also similar, but not identical, to the constrained BBO algorithm of [17].

The solution features of each BBO individual are comprised of the n elements of K, and the elements of an $n \times n$ matrix Q, where $P = QQ^T$ (see Eqs. 9 and 10). This characterization of P ensures that it is positive semidefinite for any Q matrix. We see that the optimization problem is $(n^2 + n)$-dimensional, where n is the dimension of the dynamic system. Each BBO candidate solution includes $(n^2 + n)$ solution features, which are the n elements of K and the n^2 elements of Q. For the ellipsoidal set problem, Equation (9) shows that we need to enforce $(n + k + 1)$ constraints. The first n constraints are that the n eigenvalues of $(P(A - BK) + (A - BK)^T P)$ are negative. The next k constraints are the $\Gamma_i P^{-1} \Gamma_i^T \leq 1$ inequalities. The last constraint is that $KP^{-1}K^T \leq U^2$. For each BBO individual x, we find out how many of these $(n+k+1)$ constraints are violated, and this gives us $c(x)$ in Equation (16). The cost $f(x)$ in Equation (16) is log det P.

The semi-ellipsoidal set problem is similar to the ellipsoidal set problem, as seen by comparing Eqs. 9 and 10.

The invariant cylinder problem requires a few minor BBO modifications, as seen from Eq. 15. First, we have an additional independent variable, α. Second, sliding surface coefficients are assumed to have been selected previously, so that A_w is a Hurwitz matrix, so there is no need to optimize with respect to a stabilizing feedback gain (as done for K in the ellipsoidal and semi-ellipsoidal cases). Finally, no control magnitude bounds are included in the current formulation. The BBO problem for invariant cylinder approximation is thus an (n^2+1)-dimensional problem with $(n + k)$ constraints.

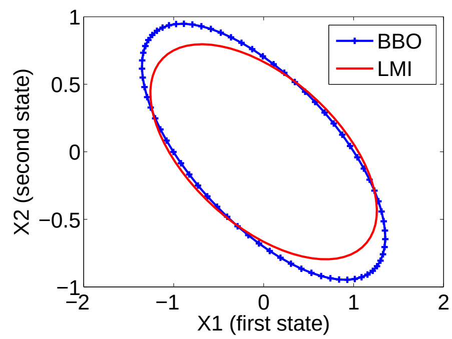

Fig. 1. Maximal ellipsoidal set approximations. The BBO algorithm finds a larger invariant set than the LMI algorithm.

IV. Simulation Results

This section presents some BBO simulation results for invariant set approximations. Simulation results were generated in MATLAB®. We ran BBO by setting the population size N to 50, the function evaluation limit to 2000, the mutation rate p_m to 1%, and the elitism parameter to 2. We implemented mutation by setting each feature of each candidate solution at each generation equal to a random number taken from a uniform distribution over the problem domain with a probability of p_m. That is, at each generation, each solution feature of each individual has a p_m chance of being replaced with a random number.

The first problem is to find the maximum ellipsoidal invariant set (Eq. 9). BBO minimizes the cost $\log \det P$ with respect to Q (recall that $P = QQ^T$) and K. Based on preliminary simulations, we set the search domain for Q and K to $[0, 2]$; that is, each candidate solution feature has the domain $[0, 2]$. The problem is defined with $U = 2$ and a double-integrator system with box constraints:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \Gamma = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1/1.5 \\ 0 & -1/1.5 \end{bmatrix}$$

The results in Fig. 1 show that a significantly larger ellipsoidal volume is achieved by searching over K, in comparison with the LMI feasibility solution of [8] that considered a constant K.

The second problem is to maximize the volume of a semi-ellipsoidal recoverable set (Eq. 10). The cost function, optimization parameters, and constants are the same as those for the first problem, and the state constraints are given by $|x_1| \leq 1$ and $|x_2| \leq 1$.

Fig. 2 compares the BBO solution with the results reported by O'Dell [13], which use a standard optimization routine (MATLAB's `fmincon`). Although BBO converges to a recoverable set with a smaller volume, the time taken by BBO to converge is much less O'Dell's algorithm (4.70 seconds vs. 7.25 seconds). Computational effort is not an issue for a two-dimensional problem, but it can become a bottleneck for high-dimensional, real-world problems.

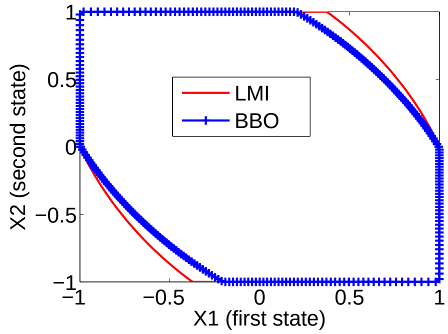

Fig. 2. Maximal semi-ellipsoidal set approximations. The LMI algorithm finds a larger invariant set, but the BBO algorithm only requires 65% of the computational effort.

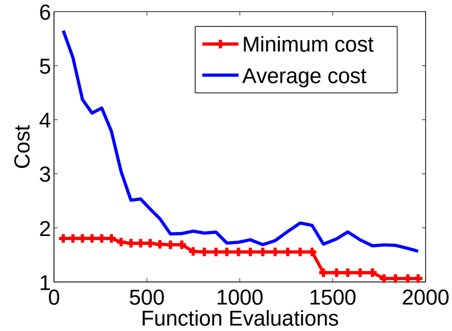

Fig. 3. BBO convergence for the maximal invariant cylinder problem. The plot shows the average cost of all 100 individuals in the BBO population, and the cost of the best individual, at each generation. It appears that the best cost might continue to improve with additional function evaluations.

	Cost	P	α
BBO	**1.032**	$\begin{bmatrix} 0.6114 & -0.0080 \\ -0.0080 & 0.5372 \end{bmatrix}$	0.3803
LMI	1.188	$\begin{bmatrix} 0.6358 & -0.0843 \\ -0.0843 & 0.5523 \end{bmatrix}$	0.4258

TABLE I

MAXIMAL INVARIANT CYLINDER APPROXIMATIONS. BBO FINDS A SOLUTION 13% BETTER THAN LMI.

The third problem is to compute the cylindrical invariant set (Eq. 15). The cost function for this problem is the same as for the previous problems, and the problem data is the same as in the example of [12]. The search domain for each element of Q is $[-2, 2]$ (recall that $P = QQ^T$) and the search domain for α is $[-\alpha_0, 0]$. Problem parameters are $B_1^T = [-0.8 \quad -0.4]$ and

$$A_w = \begin{bmatrix} -1 & -0.5 \\ 1 & -2 \end{bmatrix} \quad \Gamma = \begin{bmatrix} 0.8 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0.8 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1.25 \end{bmatrix}$$

The disturbance bound and switching gain are $\bar{\zeta} = 0.2$ and $\eta = 0.8$, respectively. The results shown in Table I indicate that the recoverable set found with BBO has a larger volume than the one found with the LMI feasibility method of [12].

Figure 3 shows typical BBO convergence results for the cylindrical invariant set problem. Recall that there are 100 candidate solutions in a BBO simulation. The best individual converges to a cost of 1.032 (also show in Table I), while the average of the 100 candidate solutions is 1.566 after 2000 function evaluations. The minimum cost is monotonically nonincreasing due to the use of elitism in BBO, but the average cost fluctuates during the optimization process.

V. CONCLUSIONS

We have proposed a constrained BBO algorithm, which is similar to, but distinct from, previously published constrained optimization algorithms. We have applied constrained BBO to three invariant set approximation problems. BBO performs better than LMI algorithms on two of the three problems. LMI performs better than BBO on one of the problems, but BBO requires only 65% of the computational effort of LMI. We terminated our BBO simulations after 20,000 function evaluations, although it appears that BBO was still improving.

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