Position and Current Control of an Interior Permanent-Magnet Synchronous Motor by Using Loop-Shaping Methodology: Blending of $H_{\infty}$ Mixed-Sensitivity Problem and T–S Fuzzy Model Scheme

This paper presents a robust mixed-sensitivity $H_{\infty}$ controller design via loop-shaping methodology for a class of multiple-input multiple-output (MIMO) uncertain nonlinear systems. In order to design this controller, the nonlinear plant is first modeled as several linear subsystems by Takagi and Sugeno’s (T–S) fuzzy approach. Both loop-shaping methodology and mixed-sensitivity problem are then introduced to formulate frequency-domain specifications. Afterward for each linear subsystem, a regional pole-placement output-feedback $H_{\infty}$ controller is employed by using linear matrix inequality (LMI) approach. The parallel distributed compensation (PDC) is then used to design the controller for the overall system. Several experimental results show that the proposed method can effectively meet the performance requirements like robustness, good load disturbance rejection, and both tracking and fast transient responses even in the presence of parameter variations and load disturbance for the three-phase interior permanent-magnet synchronous motor (IPMSM). Finally, the superiority of the proposed control scheme is approved in comparison with the input–output linearization (I/O linearization) and the $H_2/H_{\infty}$ controller methods. [DOI: 10.1115/1.4024200]

Keywords: LMI, mixed-sensitivity, loop-shaping, robust control, T–S fuzzy model, three-phase interior permanent-magnet synchronous motor (IPMSM)

1 Introduction

Interior permanent-magnet synchronous motors are extensively used in industry, due to their comparatively high efficiency, exceptional power density, and excellent torque generating. Over the last decade, there have been numerous progresses in the development of controllers for interior permanent-magnet (IPM) synchronous motors as listed below. Lin et al. [1] used a nonlinear position controller based on input–output feedback linearization technique for an IPMSM control system. Yang and Zhong [2] proposed a robust speed-tracking control of permanent-magnet synchronous motor (PMSM) servo systems to eliminate speed-tracking error due to time-varying load torque uncertainty and parameter perturbations. Su et al. [3] developed a highly robust automatic disturbance rejection controller to implement high-precision motion control of permanent-magnet synchronous motors. Chou and Liaw [4] developed robust 2-DOF current and torque control scheme for a PMSM drive with satellite reaction wheel load. Lin et al. [5] designed the adaptive back stepping proportional-integral (PI) sliding-mode controller for interior permanent-magnet synchronous motor drive systems. Azimi et al. [6] designed a robust multi-objective $H_2/H_{\infty}$ tracking controller based on T–S fuzzy model for a class of nonlinear uncertain drive systems. The available papers also employed the sliding-mode technique of PMSM or IPMSM drive systems [7,8]. Recent researches show that a T–S fuzzy model can be utilized to approximate global behavior of highly complex nonlinear systems. The large numbers of published paper have used the T–S fuzzy model technique for different drive systems [9,10].

The main contribution of this research is position and current control of an IPMSM by using loop-shaping methodology: blending of robust mixed-sensitivity $H_{\infty}$ problem and T–S fuzzy model scheme. In this paper, the problem of robust mixed-sensitivity $H_{\infty}$ control of an IPMSM system which possesses not only parameter uncertainties but also external disturbances is considered. Several robust $H_{\infty}$ loop shaping and mixed-sensitivity problem schemes based on the use of LMIs theory have also been proposed in Refs. [11–13]. In the proposed method, a Takagi–Sugeno (T–S) fuzzy model is first designed on behalf of given nonlinear plant. The fuzzy model is described by fuzzy IF–THEN rules which represent local input–output relations of a nonlinear system. Both loop-shaping methodology and mixed-sensitivity problem are then introduced to formulate frequency-domain specifications. Afterward, a regular flowchart is proposed to get optimal weighting functions. In this research, tracking of rotor angular position and d-axis current are selected as the main objectives, in the event that second objective refer to elimination of reluctance effects and torque ripple. Thereafter, a robust mixed-sensitivity $H_{\infty}$ output-feedback controller with regional pole-placement is designed by LMI-based design techniques for each linear subsystem. The PDC is then used to design the controller for the overall system. Furthermore, the overall fuzzy model of the system is achieved by fuzzy blending of the local linear system models. Eventually, several results show that the proposed method can effectively meet the performance requirements like robustness, good load disturbance...
rejection, tracking, and also fast transient responses for the three-phase IPMSM system.

The paper is organized as follows: problem statement and IPMSM dynamic model are presented in Sec. 2. In Sec. 3, T–S fuzzy model of IPMSM and problem formulation are introduced. Section 4 describes \( H_{\infty} \) loop-shaping and the mixed-sensitivity problem. In Sec. 5, design of robust tracking controller is proposed. Simulation results of the closed-loop system with the proposed technique are presented in Sec. 6 and finally, the paper is concluded in Sec. 7.

2 Problem Statement and IPMSM Dynamic Model

2.1 Problem Statement. The dynamics of IPMSM are nonlinear and may also contain uncertain parameters such as viscous damping coefficient and stator windings resistance. Consequently, the control performance of IPM synchronous motors in various applications is highly touchy to variations of external load and system parameters. Noting that these variations are noxious factors in the IPMSM control system, in this paper, an innovative robust position and current control will be presented to reduce their effects. The major contributions of this study are as follows:

- Successful employment of a proper T–S fuzzy model on behalf of original nonlinear plant.
- Successful design of feasible robust position and current controller based on a suitable T–S fuzzy model in the presence of parameters uncertainties as well as unknown load disturbance.
- Successful development of transient responses and disturbance attenuation of position and current tracking.
- Successful robustness of designed system ensuring that all closed-loop performance specifications are satisfied in the presence of unavoidable model uncertainty when the parameters in the system dynamic are varied in a wide range.
- Successful superiority of proposed strategy in comparison with former design procedures.
- Successful responses of rotor angular position control for various position commands, in order to prove to afford tracking of different position reference inputs.

2.2 IPMSM Dynamic Model. The IPMSM nonlinear dynamics in the \( d-q \) reference frame can be described by the set of equations [1,6]

\[
\begin{align*}
\frac{d \theta}{dt} &= \omega, \\
\frac{d \omega}{dt} &= \frac{3}{2} \frac{P_0}{J_m} (L_d - L_q) i_d + \theta_1 i_q - \frac{B_m}{J_m} \omega - \frac{C_1}{J_m}, \\
\frac{d i_d}{dt} &= -R_e i_d + \frac{P_0}{J_m} L_d \omega i_q + \frac{1}{L_d} v_d, \\
\frac{d i_q}{dt} &= -P_0 \frac{\theta_1}{L_q} \omega + \frac{P_0}{L_q} L_d \omega i_d + R_e i_q + \frac{1}{L_q} v_q.
\end{align*}
\]

(1)

In Eq. (1), \( \theta_1 \) is the angular position of the motor shaft, \( \omega_1 \) is the angular velocity of the motor shaft, \( i_d \) is the direct current and \( i_q \) is the quadrature current. Also, \( \theta_1 \) is the flux linkage of the permanent magnet, \( P_0 \) is the number of pole pairs, \( R_e \) is the stator windings resistance, \( L_d \) and \( L_q \) are the direct and quadrature stator inductances, respectively. Moreover, \( J_m \) is the rotor moment of inertia, \( B_m \) is the viscous damping coefficient and \( C_1 \) is the load torque. In addition, \( v_d \) is the direct voltage and \( v_q \) is the quadrature voltage. The electromagnetic torque of the motor can be described as follows:

\[
T_e = \frac{3}{2} P_0 [(L_d - L_q) i_d + \theta_1 i_q] \tag{2}
\]

The parameters \( R_e \) and \( B_m \) are supposed to differ from their nominal values \( R_{0e} \) and \( B_{0m} \). The following equation indicates a state-space representation of the synchronous motor:

\[
\begin{align*}
x_1 &= x_2 \\
x_2 &= (\eta_1 x_1 + \eta_2) x_4 + \eta_3 x_2 - \frac{C_1}{J_m} \\
x_3 &= \eta_4 x_3 + \eta_5 x_2 x_4 + \eta_6 u_1 \\
x_4 &= \eta_7 x_3 + \eta_8 x_2 + \eta_9 x_4 + \eta_{10} u_2
\end{align*}
\]

(3)

where

\[
x = [x_1 \ x_2 \ x_3 \ x_4]^T = [\theta_1 \ \omega_1 \ i_1 \ i_1]^T \tag{4}
\]

\[
[u_1 \ u_2]^T = [v_d \ v_q]^T \tag{5}
\]

In Eq. (3), the parameters \( \eta_i \) can be expressed in the following form:

\[
\begin{align*}
\eta_1 &= \frac{3}{2} \frac{P_0}{J_m} (L_d - L_q) \\
\eta_2 &= \frac{3}{2} \frac{P_0}{J_m} \theta_1 \\
\eta_3 &= -\frac{B_m}{J_m} \\
\eta_4 &= R_e \\
\eta_5 &= \frac{P_0}{L_d} L_d \\
\eta_6 &= \frac{1}{L_d} \\
\eta_7 &= -P_0 \frac{\theta_1}{L_q} \\
\eta_8 &= -P_0 \frac{L_d}{L_q} \\
\eta_9 &= -\frac{R_e}{L_q} \\
\eta_{10} &= \frac{1}{L_q}
\end{align*}
\]

3 T–S Fuzzy Model of IPMSM and Problem Formulation

In this section, the T–S fuzzy dynamic model is described by fuzzy IF–THEN rules, which represent local linear input–output relations of nonlinear systems [14,15]. The fuzzy dynamic model is proposed by Takagi and Sugeno. Over the last years, large number of authors employed T–S fuzzy approach to approximate nonlinear system [16,17]. The ith rule of T–S fuzzy dynamic model with parametric uncertainties can be described as follows:

\[
\begin{align*}
\text{IF } v_1(t) \text{ is } M_{i1} \text{ and } ... \text{ and } v_p(t) \text{ is } M_{ip} \text{ THEN } \\
x(t) &= [A_{i1} + \Delta A_{i1}] x(t) + [B_{i1} + \Delta B_{i1}] u(t) + [B_{i2} + \Delta B_{i2}] u(t)] \\
z(t) &= [C_{i1} + \Delta C_{i1}] x(t) + [D_{i11} + \Delta D_{i11}] u(t) + [D_{i12} + \Delta D_{i12}] u(t)] \\
y(t) &= [C_{i2} + \Delta C_{i2}] x(t) + [D_{i21} + \Delta D_{i21}] u(t) + [D_{i22} + \Delta D_{i22}] u(t)] \\
\end{align*}
\]

(7)

where, \( M_{ij} \) is the fuzzy set; \( r \) is the number of IF–THEN rules and \( v_j(t) \rightarrow v_j(t) \) are the premise variables; \( x(t) \in R^m \) is the state vector; \( u(t) \in R^m \) is the control input vector; \( w(t) \in R^q \) is the disturbance input vector; \( y(t) \in R^p \) is the output vector. The matrices: \( \Delta A_i, \Delta B_{i1}, \Delta C_{i1}, \Delta C_{i2}, \Delta D_{i11}, \Delta D_{i12}, \Delta D_{i21}, \Delta D_{i22} \) represent the uncertainties in the system and satisfy the following assumption:

\[
\begin{align*}
&A_{i1} = G(x(t), t) H_{i1}, \quad \Delta A_{i1} = G(x(t), t) \Delta H_{i1}, \quad \Delta B_{i1} = G(x(t), t) \Delta H_{i1}, \quad \Delta B_{i2} = G(x(t), t) \Delta H_{i2}, \\
&\Delta C_{i1} = G(x(t), t) \Delta H_{i1}, \quad \Delta D_{i11} = G(x(t), t) \Delta H_{i1}, \quad \Delta D_{i12} = G(x(t), t) \Delta H_{i1}, \quad \Delta D_{i21} = G(x(t), t) \Delta H_{i1}, \\
&\Delta D_{i22} = G(x(t), t) \Delta H_{i1}, \quad \Delta D_{i22} = G(x(t), t) \Delta H_{i1}, \quad \Delta D_{i22} = G(x(t), t) \Delta H_{i1}
\end{align*}
\]

(8)

where \( H_{ij}; j = 1, ..., 9 \) are known matrix functions which characterize the structure of the uncertainties. Furthermore, the
following inequality holds: \( G(x(t), t) \leq \partial, \partial > 0 \). Considering the nonlinear state space model (1), the parameters \( R_i \) and \( B_m \) and load torque disturbance input \( C_i \) are supposed to vary. According to the prementioned local linearization approach, the local linear model matrices for the system (3) with mentioned variations at the ith selected operating point are obtained as follows:

\[
A_i = A_{0i} + B_{m}A_{Ri} + R_iA_{K},
\]

\[
B_i = \begin{bmatrix}
0 & 0 & 0 \\
1 / J_m & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
\]

\[
B_2 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
\]

\[
A_{0i} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

where \( A_{0i}, A_{Ri}, A_{Ki}, B_1, B_2 \) are known real matrices with appropriate dimensions in nonlinear model (3). In view of the set of matrices (9)–(11), \( x \times 2 \) and \( x \times 4 \) appear only in matrix \( A_{0i} \), since the nonlinear system (1) is an affine model accordingly, \( xi \) cannot be present at \( B_i \). In other words, matrix \( A_{0i} \) is only influenced by the nonlinearity term \( x \times 2 \) and \( x \times 4 \). The overall fuzzy model can be defined as following form:

\[
\begin{align*}
\dot{x}(t) & = \sum_{i=1}^{r} \mu_i(v(t))[A_i + \Delta A_i]x(t) + [B_{1i} + \Delta B_{1i}]w(t) + [B_{2i} + \Delta B_{2i}]u(t), \quad x(0) = 0 \\
z(t) & = \sum_{i=1}^{r} \mu_i(v(t))[C_{1i} + \Delta C_{1i}]x(t) + [D_{11i} + \Delta D_{11i}]w(t) + [D_{12i} + \Delta D_{12i}]u(t) \\
y(t) & = \sum_{i=1}^{r} \mu_i(v(t))[C_{2i} + \Delta C_{2i}]x(t) + [D_{21i} + \Delta D_{21i}]w(t)[D_{22i} + \Delta D_{22i}u(t)]
\end{align*}
\]

where

\[
v(t) = [v_1(t) \ldots v_p(t)]
\]

And weighting function is

\[
\mu_i(v(t)) = \frac{\sigma_i(v(t))}{\sum_{i=1}^{p} \sigma_i(v(t))}
\]

\[
\sigma_i(v(t)) = \prod_{k=1}^{p} M_k(v_k(t))
\]

And it should be noted that

\[
\sigma_i(v(t)) \geq 0, \quad i = 1, 2, \ldots, r; \quad \sum_{i=1}^{r} \sigma_i(v(t)) > 0
\]

\[
\mu_i(v(t)) \geq 0, \quad i = 1, 2, \ldots, r; \quad \sum_{i=1}^{r} \mu_i(v(t)) = 1
\]

4 The \( H_\infty \) Loop-Shaping and the Mixed-Sensitivity Problem

Loop shaping is a design procedure to formulate frequency-domain specifications as \( H_\infty \) constraints problems [18–21]. To get a feeling for the loop-shaping methodology, consider the general pattern loop of Fig. 1.

In this figure, \( P(s) \) is the generalized plant, \( K(s) \) is the controller, \( u \) are the control signals, \( y \) are the measured variables, \( w \) are the exogenous signals and \( z \) are the main objectives. In this frequency-domain method, the design specifications are reflected as gain constraints on the various closed-loop transfer functions. Where the main closed-loop transfer functions are sensitivity function and complementary sensitivity function, as well as the gain constraints are shaping filters. These shaping objectives are then turned into uniform \( H_\infty \) bounds by means of shaping filters, and \( H_\infty \) synthesis algorithms are applied to compute an adequate controller. As a matter of fact, the \( H_\infty \) norm of a transfer function \( F(s) \) corresponds to the peak gain of the frequency response \( F(j\omega) \) that is denoted as

\[
F_\infty = \sup \sigma_{max}(F(j\omega))
\]

The optimal \( H_\infty \) control problem can be interpreted as minimizing the effect of the worst-case disturbance \( w \) on the output \( z \). The closed-loop transfer function from \( w \) to \( z \) is given by \( T_{zw}(s) \). Hence, the optimal \( H_\infty \) control seeks to minimize \( \|T_{zw}(s)\|_{\infty} \) over all stabilizing controllers \( K(s) \). Alternatively, we can specify some maximum value \( \gamma \) for the closed-loop RMS gain as

\[
F(P, K)_{H_\infty} < \gamma
\]

where \( \gamma \) is guaranteed \( H_\infty \) ratio between \( z \) and \( w \). In this case, the closed-loop transfer function \( T_{zw}(s) \) in frequency domain can be represented as follows:

**Fig. 1** The control structure
where \( S(s) \) is the sensitivity transfer matrix (the transfer function from \( r \) to \( e \)) and \( T(s) \) is the complementary sensitivity transfer matrix (the transfer function from \( r \) to \( y \)) that can be expressed as

\[
S(s) = (I + G(s)K(s))^{-1}
\]

\[
T(s) = G(s)K(s)(I + G(s)K(s))^{-1}
\]

The transfer functions \( W_{S}(s) \) and \( W_{T}(s) \) are two frequency dependent weighting functions (shaping filters), the sensitivity and the complementary sensitivity weighting functions, respectively. As it is known, shaping \( T(s) \) is desirable for tracking problems, noise attenuation and for robust stability with respect to multiplicative output uncertainties. On the other hand, since \( S(s) \) relates the error signals with references and disturbances, shaping the sensitivity function will permit the performance (in terms of command tracking and disturbance attenuation) of the system to be controlled. Accordingly, in order to realize above requirement, following close-loop transfer functions can be normalized as following forms:

\[ W_{r}(s)S(s) < 1 \quad W_{T}(s)T(s) < 1 \] (21)

Noting that the rotor angular position and the \( d \)-axis current are selected as the main objectives, as a result it can conclude that numbers of both system outputs and tracking errors are 2. Therefore, the size of weighting function matrices \( W_{S}(s) \) and \( W_{T}(s) \) are chosen \( 2 \times 2 \) that can be expressed as follows:

\[ T(s) = T_{pr} = \begin{bmatrix} T_{1} = T_{11}r_{1} \\ T_{2} = T_{12}r_{2} \end{bmatrix}, \quad S(s) = T_{er} = \begin{bmatrix} S_{1} = T_{e1}r_{1} \\ S_{2} = T_{e2}r_{2} \end{bmatrix} \] (22)

where \( r_{1} \) and \( r_{2} \) are angular position and \( d \)-axis current; \( r_{1} \) and \( r_{2} \) are position command and direct current reference input; \( e_{1} \) and \( e_{2} \) are the tracking errors. The sensitivity and the complementary sensitivity weighting functions \( W_{T}(s), W_{S}(s) \) are designed as two \( 2 \times 2 \) diagonal matrices. Where \( W_{T}(s) \) and \( W_{S}(s) \) are the diagonal elements of \( W_{T}(s), W_{S}(s) \), respectively, that they can be rewritten as follows:

\[ W_{T}(s) = W_{T_{pr}}(s)I_{2 \times 2}, \quad W_{S}(s) = W_{S_{pr}}(s)I_{2 \times 2} \] (23)

The transfer functions \( W_{S}(s) \) and \( W_{T}(s) \) must be stable, minimum phase and additionally should be proper. As well as \( W_{S}(s) \) and \( W_{T}(s) \) must be low-pass and high-pass filters, respectively. In order to determine the performance and robustness weights, practical formulas of \( W_{S}(s) \) and \( W_{T}(s) \) are suggested as follows:

\[ W_{r}(s) = \frac{ds + 1}{ds + 2}, \quad W_{S}(s) = \frac{as + wc}{s + wc} \] (24)

where “\( a \)” high frequency disturbances gain, “\( b \)” is the gain for low frequency control signal, “\( d \)” is a constant and “\( wc \)” is the crossover frequency. \( wc \) is the frequency which differentiates the high frequency disturbance signal and the low frequency control signal. Also, \( wc \) indicates the minimum bandwidth of the transfer functions which are weighted. In order to design the optimal weighting functions, the regular weights design method (RWDM) can be expressed as a flowchart in Table 1.

Consequently, RWDM selects optimal weighting functions in order to formulate performance and robustness specifications of close-loop system. In the RWDM, the coefficients \( a, b, c, wc \) should be determined or incremented as far as whole of clauses in each step are realized and finally controller \( K \) is returned.

### 5 Design of Robust Tracking Controller

In this section, focus to design a local pole-placement output-feedback controller for each linear subsystem

\[
\text{IF } r_{i}(t) = M_{i1} \text{ and } \ldots \text{ and } r_{p}(t) = M_{rp} \text{ THEN } u(t) = K_{iy}(t), \quad i = 1, 2, \ldots, r
\] (25)

where \( K_{i} \) (\( i = 1, 2, \ldots, r \)) are the local controller gains. In order to design whole controller, the concept of PDC is employed for the system (7) [14,17]. According to PDC approach, the control law of the whole system is the weighted sum of the local feedback control of the various subsystems that can be expressed as follows:

\[
u(t) = \sum_{j=1}^{r} \mu_{j}K_{j}y(t)
\] (26)

where the local pole-placement output-feedback gains \( K_{j} \) are determined by LMI-based design techniques in order to achieve the design requirements [22,23]. The LMI formulation is applicable to design local controllers that are introduced as follows.

### Table 1 RWDM

<table>
<thead>
<tr>
<th>Answer: NO</th>
<th>Step number</th>
<th>Step description</th>
<th>Answer: YES</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>Get ith local linear subsystem, ( G_{i} ) and new input values: ( a, b, c, wc )</td>
<td>Go to next step</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>Design weighting matrices, Build the plant ( P ) and merge ( G_{i}, W_{T}(s), W_{S}(s) )</td>
<td>Go to next step</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>Specify LMI pole-placement region, ( D ) and Find local controller by LMI approach, ( K )</td>
<td>Go to next step</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>Find the close-loop responses: ( s(s): [S_{1}(s), S_{2}(s)] ) and ( T(s): [T_{1}(s), T_{2}(s)] )</td>
<td>Go to next step</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>Are these two constraints?</td>
<td>Go to next step</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \sigma_{\max}(S_{i}(j\omega)) &lt; \gamma \sigma_{\max}(W_{T_{i}}(j\omega)) ) ( \sigma_{\max}(T_{i}(j\omega)) &lt; \gamma \sigma_{\max}(W_{S_{i}}(j\omega)) )</td>
<td>Go to next step</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>Go back to: Step 2 and/or Step 5</td>
<td>Go to next step</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>Is the guaranteed ( H_{\omega} ) less than 1?</td>
<td>Go to next step</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Is close-loop system steady state and transient responses expected for tracking and disturbance attenuation</td>
<td>Go to next step</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( K ) is optimal and ( W_{T}(s), W_{S}(s) ) are desirable</td>
<td>Go to next step</td>
</tr>
</tbody>
</table>

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Main objective is to design an output-feedback controller $u = Ky$ as follows:

- It maintains the $H_\infty$ norm of $T_\infty(s)$ (RMS gain) below some prescribed value $\gamma > 0$.
- It maintains the $H_2$ norm of $T_2(s)$ (LQG cost) below some prescribed value $\nu > 0$.
- It places the closed-loop poles in some prescribed LMI region $D$.
- It minimizes a trade-off criterion of the form

$$\|P(s)z\|_\infty + \beta\|T_2\|_2$$

where $T_\infty(s)$ and $T_2(s)$ are the closed-loop transfer functions from $w$ to $z_\infty$ and from $w$ to $z_2$, respectively. The linear fuzzy subplant $P(s)$ for the control structure Fig. 1 is given in state-space form by

$$\begin{align*}
\dot{x} &= Ax + B_1 w + B_2 u \\
z &= C_2 x + D_{21} w + D_{22} u \\
y &= C_3 x + D_{31} w
\end{align*}$$

(27)

And related controller $K(s)$ is introduced as

$$\begin{align*}
\dot{z} &= A_1 z + B_1 y \\
u &= C_1 z + D_{12} w
\end{align*}$$

(28)

By merging $P(s), K(s)$, and $u = Ky$, the closed-loop system is concluded as following form:

$$\begin{align*}
\dot{x}_d &= A_{d1} x_d + B_{d1} w \\
z_\infty &= C_{d1} x_d + D_{d1} w \\
z_2 &= C_{d2} x_d + D_{d2} w
\end{align*}$$

(29)

Our three design objectives can be expressed as follows:

- **$H_\infty$ performance**: The closed-loop RMS gain from $w$ to $z_\infty$ does not exceed $\gamma > 0$ if and only if there exists a symmetric matrix $X_\infty$ such that

$$\begin{bmatrix}
A_d X_\infty + X_\infty A_d^T & B_d & X_\infty C_{d1}^T \\
C_{d1} X_\infty & -I & D_{d1}^T \\
B_{d1} & D_{d1} & -^\gamma I
\end{bmatrix} < 0, X_\infty > 0$$

(30)

- **$H_2$ performance**: The $H_2$ norm of the closed-loop transfer function from $w$ to $z_2$ does not exceed $\nu > 0$ if and only if $D_{d2} = 0$ and there exist two symmetric matrices $X_2$ and $Q$ such that

$$\begin{bmatrix}
Q & C_{d2} X_2 \\
X_2 C_{d2}^T & X_2
\end{bmatrix} > 0$$

and

$$\begin{bmatrix}
A_d X_2 + X_2 A_d^T & B_d \\
B_{d1} & -I
\end{bmatrix} < 0$$

(31)

- **Pole placement**: The closed-loop poles lie in the LMI region $D$ if and only if there exists a symmetric matrix $X_{pol}$ satisfying

$$\begin{bmatrix}
\mu_j (A + B_2 K) X_{pol} + \mu_j X_{pol} + \mu_j X_{pol} (A + B_2 K)^T
\end{bmatrix} < 0$$

(33)

For tractability in the LMI framework, we seek a single Lyapunov function $X = X_\infty = X_2 = X_{pol}$ that enforces all three sets of constraints. Factorizing $X$ as

$X = X_1 X_2^T$

And introducing the change of controller variables

$$\begin{align*}
A_k := N A_3 M^T + N B_3 C_1 R + S B_3 C_3 M^T + S (A + B_2 D_4 C_4) R \\
B_k := N B_k + S B_2 D_k \\
C_k := C_4 M^T + D_k C_3 R
\end{align*}$$

(34)

The inequality constraints on $\gamma$ are readily turned into LMI constraints in the variables $R, S, Q, A, B, C, K$, and $D$. This leads to the following suboptimal LMI formulation of our multi-objective synthesis problem:

Minimize $\gamma^2 + \beta \text{Trace}(Q)$ over $R, S, Q, A, B, C, K, D$, and $\gamma^2$ satisfying:

$$\begin{bmatrix}
AR + RA^T + B_2 C_K & C_3^T B_T^2 & A_K^T + A + B_2 D_4 C_4 & B_1 + B_2 D_4 D_3 \\
H & A^T S + S A + B_k C_4 + C_3^T B_T^2 & B_1 + B_2 D_4 D_3 \\
C_4 R + D_{22} C_K & C_3 + D_{22} D_4 C_4 & D_{22} + D_{22} D_4 D_3 \\
\gamma^2 I
\end{bmatrix} < 0$$

(35)

$$\begin{bmatrix}
\lambda_j (R I) & \mu_j (AR + B_2 C_K + A + B_2 D_4 C_4) \\
\mu_j (A_K + S A + B_k C_4) & A_K^T + A + B_2 D_4 C_4 \\
\gamma^2 I & \mu_j (R A^T + C_3^T B_T^2 & A_K^T + A + B_2 D_4 C_4)
\end{bmatrix} < 0$$

(36)

where $Q$ is a symmetric matrix as the sum of the elements on the main diagonal of $Q$ must be less than $\nu_0, \gamma_0$ and $\nu_0$ are prescribed values, as $H_\infty$ norm of $T_\infty(s)$ ($T_{\infty, w}$) and $H_2$ norm of $T_2(s)$ ($T_{\infty, w}$) lie below of them, respectively. Given optimal solutions $\gamma^*, Q^*$ of this LMI problem, the closed-loop $H_\infty$ and $H_2$ performances are bounded by
The main goal of this research is design of a proper control which guarantees robust performance in presence of parameters variation as well as load torque disturbance. In this case, two control objectives are defined. First, rotor angular position and second, the $d$-axis current of motor that both of them must track reference trajectories $r_1$ and $r_2$, respectively. In fact, $d$-axis current must be converged to zero, due to elimination of reluctance effects and torque ripple. Therefore, in order to achieve these objectives, both of tracking errors should be minimized. Mentioned goals are realized through constructing of the objectives $z$ in an appropriate control loop. Under the above considerations, the structure of the mixed-sensitivity control loop is shown schematically in Fig. 2.

According to this structure, the nonlinear dynamic is first approximated with some local linear models in four rules which are represented by T–S fuzzy approach. Both shaping filters $W_z(s)$ and $W_p(s)$ are then designed and augmented plant $P$ is created. Thereafter, local controllers are designed for each linear subplant $p_i$ based on LMI approach. After that the total augmented linear system $P$ is obtained by using the weighted sum of the local linear subsystems and is utilized rather than original nonlinear system. Moreover, according to PDC approach, the control law of the whole system $K$ is designed. On the other hand, fuzzy weights ($\mu_i$) that are shown in Fig. 2 are updated by using a fuzzy weights online computation component that is shown in Fig. 3.

- In the first block of this component, angular velocity of the motor shaft ($x_2=\omega_r$) and the quadrature current ($x_4=i_q$) are measured in real time from IPMSM and nonlinear terms (fuzzy variables: $v_1=x_2$, $v_2=x_4$) are then built by these measurements.
- In the second block of FWOC component, the values of membership functions ($M_i$) in current values of $x_2$, $x_4$ “nonlinearities” are calculated.
In the third block, new fuzzy weights (14)–(16) are calculated and they are sent to main control structure (Fig. 2). Finally, by using whole system and global controller, a tracking loop (Fig. 2) is applied to the system in order to achieve desirable specifications such as tracking performance, bandwidth, disturbance rejection, and robustness for the close-loop system.

6 Simulation Results

In this section, we show the effectiveness of the proposed method by performing some simulation studies over an IPMSM model. The motor used in the paper is made by Hsin-Ting Company, Taiwan. The type of motor is the 130-750MS-ZK-L2. This IPMSM is a three-phase, four poles, rated 0.75 HP, with a 2000 rpm rated speed. In the experimentation, the maximum voltage and the continuous rated armature current are set to 230 V and 12 A [1, 6]. The parameters of the IPMSM are shown in Table 2.

In this case, the stator windings resistance \( R_s \) and the viscous damping coefficient \( B_m \) are varied between 650% and load torque disturbance is unknown [1, 6, 7].

According to Eq. (11), the \( x_2 \) and \( x_4 \) are nonlinear terms and referring to the IPMSM characteristic and the territory of the system operating points, we should calculate the minimum and maximum values of \( x_2 \) and \( x_4 \) as: \( v_1(t) = x_2 \in [0, 2000] \) and \( v_2(t) = x_4 \in [0, 12] \). Where \( v_1 \) and \( v_2 \) are fuzzy variables. In view of above limitations, membership functions can be calculated as

\[
M_1 = \frac{v_1}{2000}, \quad M_2 = 1 - \frac{v_1}{2000}, \quad M_3 = \frac{v_2}{12}, \quad M_4 = 1 - \frac{v_2}{12}
\]

In view of the above equations, the membership functions \( M_i \) is demonstrated in Fig. 4.

In the first step, the system (3) is represented by a T–S fuzzy model using the fuzzy rules given in Eq. (7). For this design problem, the rules \( r_1–r_4 \) are constructed for the T–S fuzzy model and referring to Eqs. (4)–(6) the four linear subsystems are then calculated. Referring to Eqs. (9)–(11), all system matrices are constant except \( A_0 \) that varies with respect to each rule as given below

- In the second step, referring to Sec. 4, the weighting matrices \( W_T(s) \) and \( W_i(s) \) are designed as follows:

\[
W_T(s) = W_{T_0}(s)I_{2 \times 2} = \frac{0.001s + 1}{0.001s + 0.02}I_{2 \times 2}
\]

\[
W_i(s) = W_{i_0}(s)I_{2 \times 2} = \frac{0.5s + 10}{s + 0.01}I_{2 \times 2}
\]

\[
A_{0i} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & -1145 & 1860 \\
0 & 49 & 0 & 0 \\
0 & -20 & -82119 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1860 \\
0 & 0 & 0 & 0 \\
0 & -20 & -82119 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1145 & 1860 \\
0 & 49 & 0 & 0 \\
0 & -20 & 0 & 0
\end{bmatrix}
\]

\[
A_{0i} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1860 \\
0 & 0 & 0 & 0 \\
0 & -20 & 0 & 0
\end{bmatrix}
\]

\[
A_{0i} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
A_{0i} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1860 \\
0 & 0 & 0 & 0 \\
0 & -20 & 0 & 0
\end{bmatrix}
\]

\[
A_{0i} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
In the third step, by using the LMI formulations fully presented in Sec. 5, we can calculate the local controllers for each linear subsystem based on purposed control loop (Fig. 2), above weighting matrices and LMI technique (27)–(36).

In order to design state-space feedback gains \( K_i \) for each subsystem, the following steps are done:

1. Specify the LMI region \( \mathcal{D} \) (32), in order to place the closed-loop poles in this region (pole placement) and also to guarantee some minimum decay rate and closed-loop damping. The aforementioned region is shown in Fig. 5, as: the intersection of the half-plane is \( x < -8 \) and center of sector is at the origin with inner angle \( \pi/2 \). \( \mathcal{D}_1 : x < -8 \) and \( \theta = \pi/2 \).

2. Choose a four-entry vector specifying the \( H_2/H_{\infty} \) trade-off criterion: \([\gamma_0 \nu_0 \beta \bar{\beta}] = [0 \ 0 \ 1 \ 0] \). As a matter of fact, in this case, constraint and criterion of \( H_2 \) is not used (\( \bar{\beta} = 0 \)) and two design objectives, \( H_{\infty} \) performance and pole placement are only employed.

3. Minimize \( H_2/H_{\infty} \) cost function subject to the aforementioned pole-placement constraint using Eqs. (30), (31), and (34)–(36).

Finally, the overall fuzzy system is obtained by using a weighted average defuzzifier and also the control law of the whole system is designed by using the PDC approach. Global proposed T–S fuzzy model can represents the original nonlinear system in the prespecified domains \([0, 2000] \) rpm \( \times [0, 12] \) A on the \( x_2-x_4 \) space for various operating point.

Figure 6 compares states of the proposed T–S fuzzy model in comparison with the original nonlinear system \( \chi(t) \) and the dashed lines indicate states of original nonlinear system \( \chi(t) \) in

---

**Fig. 6** Time responses of the proposed model (solid) and the original nonlinear model (dashed): (a) Angular position of motor shaft, (b) angular speed, (c) \( \delta \)-axis current, and (d) \( \varphi \)-axis current

**Fig. 7** Load torque

**Fig. 8** Disturbance rejection on angular position (a) with step load torque (1 Nm) and (b) with benchmark load torque (Fig. 5)
Table 3 The position control performance

<table>
<thead>
<tr>
<th>Method</th>
<th>Proposed $H_2/H_\infty$ (represented in Ref. [6])</th>
<th>I/O linearization (represented in Ref. [1])</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_p$ (s)</td>
<td>0.15</td>
<td>0.28</td>
</tr>
<tr>
<td>$E_{ip}$ (deg)</td>
<td>0.06</td>
<td>0.2</td>
</tr>
<tr>
<td>$E_{ip-p}$ (deg)</td>
<td>0.3</td>
<td>0.75</td>
</tr>
</tbody>
</table>

Fig. 9 $d$-axis current

Fig. 10 position tracking responses at different position reference

Fig. 11 Comparison of transient responses (a) certain step position and (b) rectangular position command

$$E_{ip-p}$$

which means that the fuzzy model can represent the original system in the prespecified domains with suitable approximation. On the other word, the proposed fuzzy model can be replaced on behalf of given original system (1) in the prespecified domains.

In actuality, the motor is used to convert the electrical energy into mechanical energy. Accordingly, an external load is added to the drive system. The external load can be selected one of two types of following disturbances: First, another synchronous motor is coupled to the shaft of the main IPMSM motor in order to request a load torque $[6,7]$. The manners of the load torque $C_i$ applied to the synchronous motor are represented in Fig. 7. Second, a 1 kg weight is located at a certain location of motor. As a result, the weight can provide the external load 1 Nm that interpreted with step input.

Figures 8(a) and 8(b) demonstrate disturbance rejection of angular position over three different methods, when step load torque and benchmark load torque (Fig. 7) act on system, respectively; although the IPMSM is first controlled to reach a fixed position “90 deg” by proposed controller. The manners of the load torque $C_i$ applied to the synchronous motor are represented in Fig. 7. Second, a 1 kg weight is located at a certain location of motor. As a result, the weight can provide the external load 1 Nm that interpreted with step input.

Table 4 Performance of proposed method for various LMI regions

<table>
<thead>
<tr>
<th>LMI region</th>
<th>$D_1$</th>
<th>$D_2$</th>
<th>$D_3$</th>
<th>$D_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Item</td>
<td>$x &lt; -8 \theta = \pi/2$</td>
<td>$x &lt; -8 \theta = 2\pi/3$</td>
<td>$x &lt; 0 \theta = \pi/2$</td>
<td>$x &lt; 0 \theta = 2\pi/3$</td>
</tr>
<tr>
<td>$T_p$ (s)</td>
<td>0.15</td>
<td>0.15</td>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td>$E_{ip}$ (deg)</td>
<td>0.06</td>
<td>0.1</td>
<td>0.06</td>
<td>0.1</td>
</tr>
<tr>
<td>$E_{ip-p}$ (deg)</td>
<td>0.3</td>
<td>0.7</td>
<td>0.3</td>
<td>0.7</td>
</tr>
</tbody>
</table>

As you can view, the proposed method has better disturbance attenuation in both types of external loads.

Table 3 abridges disturbance rejection performances when system in influenced by both kind of disturbances. According to this table, the proposed method has the smallest peak and peak-to-peak error values ($E_{ip}$ and $E_{ip-p}$) in comparison with other methods. Indeed, in view of the comparisons between three methods, it
In fact, numerical results of Table 3 are fascinated by the aforementioned LMI Region \( D_1 : x < -8 \) and \( \theta = \pi/2 \). Table 4 presents obtained values of settling time \( T_{\mu} \) and peak-to-peak \( E_{ip-p} \) error for various LMI regions (pole-placement regions) over the position tracking response of the proposed method.

Figure 12 illustrates the position responses, when both the stator windings resistance \( R_s \) and the viscous damping coefficient \( B_s \) are varied between \( \pm 50\% \). As you can see, the system has good robustness when the parameters in the systems dynamic are varied in a wide range.

Table 4 proves that \( D_1 \) is the optimal region among regions of this table, because all of settling time, peak and peak-to-peak errors based on this region have the smallest values in comparison with other regions. In fact, values of errors are increased when the inner angle \( \theta \) is expanded, as well as settling time is raised when the intersection of the half-plane \( x \) is reduced. Therefore, we have used design results based on the optimal LMI Region \( D_1 \).

Figures 13(a) and 13(b) demonstrate \( W_{r}^{-1} \) and \( W_{e}^{-1} \), that are greater than \( S_1, S_2 \) and \( T_1, T_2 \) over frequency domain, respectively. Furthermore, the weights \( W_{r}^{-1} \) and \( W_{e}^{-1} \) also are greater than variations of sensitivity functions and complementary sensitivity functions. Indeed, these variations are occurred over \( S \) and \( T \) due to parameters uncertainties.

7 Conclusions

In this paper, a robust mixed-sensitivity \( H_{\infty} \) controller has been designed to tracking and disturbance attenuation of position and current for an MIMO nonlinear uncertain T–S fuzzy model of IPMSM systems. An LMI approach has been developed and a mixed-sensitivity problem has been given to formulate frequency-domain specifications and achieve the robustness against nonlinear system uncertainties. The numerical results on IPMSM showed that the robust control system has small position and current tracking errors and has desired robustness against load torque disturbance and parameter variations. In addition, the superiority of the proposed control scheme was approved through simulations in compared with the input–output linearization and the \( H_{2}/H_{\infty} \) methods (represented in Refs. [1,6], respectively). Referring to Table 3, the proposed method had smallest settling time on position tracking response and the lowest amount of undershoot on disturbance rejection response. The major achievements of this
research are delivered as follows: (i) The performance requirements like good load disturbance rejection, tracking and fast transient responses in the proposed method were better than those of the other methods, (ii) The proposed method had satisfactory tracking for various angular position commands, (iii) The proposed controller had good robustness when parameter trajectories of $R_d(t)$ and $B_d(t)$ were changed in real time, and (iv) direct current of system can be tracked to 0 A for elimination of reluctance effects and torque ripple.

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References


