LMI Based Model Order Reduction Considering the Minimum Phase Characteristic of the System

Gholamreza Khademi, Haniyeh Mohammadi, and Maryam Dehghani
School of Electrical and Computer Engineering
Shiraz University
Shiraz, Iran
khademi.gh@gmail.com, h.mohammadi11668@gmail.com, mdehghani@shirazu.ac.ir

Abstract—One usual method to solve the model order reduction problem is to minimize the \( H_\infty \)-norm of the difference between the transfer function of the original system and the reduced one. In many papers, the minimization problem is solved using the Linear Matrix Inequality (LMI) approach. This paper deals with defining an extra matrix inequality constraint to guaranty that the minimum phase characteristic of the system preserves after order reduction. To overcome this, poles and zeros of the reduced system transfer function must be at Left-Half Plane (LHP). It is very easy to apply a LMI condition to force the poles of the system to be at LHP. However, the same cannot be applied to zeros easily. Thus, a special state-space realization of the system is introduced in a way to apply conditions on zeros of the reduced system. The method is applied to some sample example and the simulation results verify the performance of the proposed method.

Keywords—\( H_\infty \)-norm; Linear Matrix Inequalities (LMIs); Minimum phase system; Model order reduction.

I. INTRODUCTION

Model order reduction has been an attractive research area in recent years. The motivation of such considerable interest is the need for model order reduction in different fields of control such as simulation, identification, control system design and so on. Actually, a high order plant imposes a lot of complexity in control system design in terms of hardware, memory and implementation. It is crucial to notice the main characteristics of the system such as frequency response, stability and minimum phase property must preserve after order reduction.

Numerous methods of finding reduced order models have been introduced over the past few decades. Various methods of model order reduction are based on the \( H_\infty \) norm and the most popular methods, namely the balanced truncation methods and the optimal Hankel norm approximation provide constructive techniques for model order reduction. These methods deal with the minimization of the \( H_\infty \) norm of the difference between the main and the reduced order transfer function [1-2]. Some papers try to consider specific characteristics of a system in model order reduction. For example in [3], model reduction is done by balanced realization method and a modification is suggested that results in a better approximation of the low frequency behavior of the original system.

In the optimal Hankel norm reduction method, the \( L_2 \) gain of Hankel operator \( H_G \) never exceeds \( \| G(j\omega) \|_\infty \). It is possible to show that the upper and lower bounds of the \( H_\infty \)-norm have been obtained in terms of the Hankel singular values [2, 4-5]. Although the Hankel norm reduction method has a well-developed theory and has proved to be reliable, it is not well suited for control performance [6]. \( H_2 \) and \( H_\infty \)-norm approximations are well adapted for model reduction problems. A well-known method to solve the problem is based on Linear Matrix Inequality (LMI). A significant advantage of LMI approach is the possibility to solve model order reduction subject to additional constraints. This could mean finding reduced system subject to a pole region constraint and subject to a zero region constraint to keep the minimum phase characteristic of the original system.

The \( H_2 \) and \( H_\infty \) model order reduction problems result in bilinear matrix inequalities (BMIs). In most papers, an iterative LMI approach (ILMI) is used to solve the problem. However, no one can ignore the drawback of iterative LMI scheme due to the need for priori knowledge of decision variables.

In [6], frequency weighted \( H_2 \) norm model reduction is investigated. The problem results in the BMIs. Thus, a two-step iterative LMI method is considered as the solution and different algorithms are proposed to prevent getting stuck in a local minimum. In [7], a two-step iterative LMI scheme is used to obtain the \( H_\infty \) model error and the results are compared to Hankel norm reduction (HNR) method. It is shown that in the model reduction of a system of order \( n \) to \( k \), in case \( k = n - 1 \), the Hankel norm reduction method is optimal. In other cases, the \( H_\infty \) norm is bounded by the sum of the \( n - 1 \) smallest Hankel singular values. To resolve the drawback of iterative LMI methods, in [8] a new non-iterative \( H_\infty \)-based model order reduction using LMI approach is suggested.

All the above mentioned papers try to minimize \( \| G(s) - \hat{G}(s) \| \). However, they do not consider the minimum phase characteristic of the reduced order system. There are some papers such as [7], in which, the main system is minimum phase while the reduced order system is non-minimum phase.
Some papers try to reduce the order of the plant and design a suitable controller for it. Since designing controller for non-minimum phase systems encounters many problems, they need to reduce the plant to a minimum phase system. In [9] it is desired to design an adaptive robust backstepping controller for a hard disk drive. The backstepping controller design procedure has a limitation in stabilization of systems with non-minimum phase characteristic. Therefore, the suggested solution is to reduce the system to a minimum phase one and then applying the controller. In the above mentioned paper, the Hankel norm reduction method is used many times to achieve a reduced order minimum phase system through a trial and error approach.

The key point of the present paper is to propose a matrix inequality constraint to ensure that the minimum phase property of the main system is kept. Since undershoot and time-delay are features of non-minimum phase systems which are not desirable in controller synthesis procedures, this paper exploits a special state-space realization to preserve this characteristic.

The paper is outlined as follows. In section II, LMI based model order reduction problem is stated and the state-space realization is written in a way to apply the proposed constraint to the minimization problem. The algorithm is explained in detail in section III. In section IV, simulation results are shown. Finally, conclusions are discussed in section V.

II. MODEL ORDER REDUCTION

The main purpose of model order reduction problem is to reduce a system of nth-order with transfer function $G(s) = \frac{b_ms^n + b_{m-1}s^{n-1} + \ldots + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_0} = [A \ B] C \ D$ with lower order ($n' < n$) to keep the frequency response of these two systems close to each other as possible. Indeed, the concept of model reduction is to remove the states from $G(s)$ that are of little effect on the system input-output characteristic [10]. The problem can be considered as an optimization problem to minimize the difference between $G(s)$ and $\hat{G}(s)$ by using [11]:

$$\|\hat{G}(j\omega)\|_\infty = \|\hat{G}(j\omega) - G(j\omega)\|_\infty = \sup_{\omega} \sigma_{\text{max}}(\hat{G}(j\omega) - G(j\omega))$$

(1)

Different methods are exploited to solve the above minimization problem such as well-known Hankel norm model order reduction [1] or genetic algorithm method [12]. In this paper, linear matrix inequalities are used. Therefore, a brief description of the method is given in the following.

A. LMI Based Model Order Reduction

The problem is to find the reduced order model such that the $H_\infty$-norm of the error system is minimized. Assume the main system transfer function and the state space representation as follows [13]:

$$G(s) = \frac{b_ms^n + b_{m-1}s^{n-1} + \ldots + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_0} = [A \ B] C \ D$$

(2)

If the reduced order system is defined as $\hat{G}(s)$ with the following representation:

$$\hat{G}(s) = \frac{\hat{b}_ms^{n'} + \hat{b}_{m'-1}s^{n'-1} + \ldots + \hat{b}_0}{s^{n'} + \hat{a}_{n'-1}s^{n'-1} + \ldots + \hat{a}_0} = [\hat{A} \ \hat{B}] \hat{C} \ D$$

(3)

Where $n' < n$. The block diagram of this problem is shown in Fig.1, and the error system transfer function can be written as follows [7]:

$$\hat{G}(s) = G(s) - \hat{G}(s) = \left[ \begin{array}{cc} A & 0 \\ 0 & \hat{A} \end{array} \right] \left[ \begin{array}{c} B \\ \hat{B} \end{array} \right]$$

$$= \hat{C}(sI - \hat{A})^{-1}\hat{B} + \hat{D}$$

(4)

The goal of model order reduction is to find $\hat{A}, \hat{B}, \hat{C}$ and $\hat{D}$ such that the infinity norm of the error system is minimized. It can be shown that $\|G\|_\infty < \gamma$ is equivalent to the existence of a symmetric, positive definite matrix $P > 0$ satisfying:

$$\left[ \begin{array}{ccc} PA + A^TP & PB & C^T \\ B^TP & -\gamma I & D^T \\ C & D & -\gamma I \end{array} \right] < 0$$

(5)

This inequality is called the bounded real lemma and proposed in [14-16].

In previous model reduction researches, the $H_\infty$ norm of the error system is minimized by (5) and the best reduced order model is developed. However, the minimum phase characteristic of the reduced system is not regarded. In this paper, a constraint is added to the above minimization problem to guaranty that minimum phase characteristic of the system is kept unchanged after model order reduction. To achieve this, a special state space representation is exploited which is explained in the next sub-section.

![Fig.1. Error system transfer function](image-url)

B. LMI Constraint to Preserve the Minimum Phase Characteristic of the System

To preserve the minimum phase property of the main system, poles and zeros of the reduced system transfer function must be at Left-Hand Plane (LHP). Thus, the state-
space representation has to define in a way to be able to apply condition on the zeros of the reduced system. The main idea of the following state space realization is derived from [17].

Assume a linear system represented by the transfer function:

\[ G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_0} = \frac{N(s)}{D(s)} \tag{6} \]

Where \( \text{deg} D = n \), \( \text{deg} N = m < n \), and \( b_m \neq 0 \). The relative degree \( r = n - m \).

By Euclidean division, \( D(s) \) can be written as:

\[ D(s) = Q(s)N(s) + R(s) \tag{7} \]

Where \( R(s) \) and \( Q(s) \) are remainder polynomial and the quotient, respectively. It is obvious that the first coefficient of \( Q(s) \) is \( 1/b_m \). From (7), we know that:

\[ \text{deg} Q = n - m = r \quad \& \quad \text{deg} R < m \tag{8} \]

We can rewrite \( G(s) \) as follows:

\[ G(s) = \frac{N(s)}{Q(s)N(s) + R(s)} = \frac{1}{Q(s)} \frac{1}{1 + \frac{R(s)}{Q(s)N(s)}} \tag{9} \]

Thus, \( G(s) \) can be represented as a closed loop system shown in Fig.2. The \( r \text{th} \) order transfer function \( 1/Q(s) \) has no zeros, and can be realized by the \( r \text{th} \)-order state vector;

\[ \dot{\xi} = [y, \dot{y}, ..., y^{r-1}]^T \tag{10} \]

Therefore, the state space model for \( 1/Q(s) \) will be:

\[ \dot{\xi} = (A_c + B_c A^T) \xi + B_c b_m e \]

\[ y = C_c \xi \tag{11} \]

Where \((A_c, B_c, C_c)\) is a canonical form representation of a chain of \( r \) integrators, that is:

\[ A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 & 1 \end{bmatrix}, \tag{12} \]

\[ B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{r \times 1}, \tag{13} \]

\[ C_c = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}_{1 \times r} \tag{14} \]

\[ \lambda \in \mathbb{R}^{r \times 1} \tag{15} \]

The minimal realization of \( R(s)/N(s) \) transfer function can be written as:

\[ \eta = A_0 \eta + B_0 C_c \xi \]

\[ w = C_0 \eta \tag{16} \]

Where \( \eta \in \mathbb{R}^{m \times 1} \), \( A_0 \in \mathbb{R}^{m \times m} \), \( B_0 \in \mathbb{R}^{m \times 1} \) and \( C_0 \in \mathbb{R}^{1 \times m} \).

The eigenvalues of \( A_0 \) are the zeros of the polynomial \( N(s) \), which are the zeros of the transfer function \( G(s) \). Considering \( e = u - w \) and substituting (16) into (11), we can write state-space model of \( G(s) \) as:

\[ \dot{\eta} = A_0 \eta + B_0 C_c \xi \]

\[ \dot{\xi} = A_c \xi + B_c A^T \xi - B_c b_m C_c \eta + B_c b_m u \]

\[ y = C_c \xi \tag{17} \]

Where \( \eta \in \mathbb{R}^{m \times 1} \), \( \xi \in \mathbb{R}^{r \times 1} \).

Finally, a new representation of \( G(s) = C(sI - A)^{-1}B + D \) is defined where the state-space matrices are:

\[ A = \begin{bmatrix} A_0 & B_0 C_c \\ -B_c b_m C_0 & A_c + B_c A^T \end{bmatrix} \]

\[ B = \begin{bmatrix} 0 \\ b_m B_c \end{bmatrix} \tag{18} \]

\[ C = \begin{bmatrix} 0 & C_c \end{bmatrix} \]

\[ D = 0 \]

Since the eigenvalues of \( A_0 \) are the zeros of the transfer function \( G(s) \), it is needed to apply the following matrix inequality to be sure that the system zeros are at the LHP.

\[ P > 0, A_0^T P + PA_0 < 0 \tag{19} \]

And the following matrix inequality guaranties the stability of the system by putting the system poles at the LHP.

\[ K > 0, A^T K + KA < 0 \tag{20} \]

III. MODEL ORDER REDUCTION ALGORITHM CONSIDERING THE MINIMUM PHASE CHARACTERISTIC OF THE SYSTEM

Suppose that the main system is \( G(s) \) and the reduced order model is defined by \( \hat{G}(s) \). In addition, the state space representations of the both systems are based on (17). As it is stated before, the model order reduction problem can be considered as an optimization problem i.e. minimize \( \| \hat{G} - G \|_\infty \) with respect to \( \hat{G} \).

Consequently, the problem can be considered as a minimization problem given below:
Find the smallest possible $\gamma$ with respect to $(\hat{A}_0, \hat{B}_0, \hat{C}_0, \hat{A})$, such that $P > 0$, $P_0 > 0$, $K > 0$ and:

$$
\begin{bmatrix}
P \hat{A} + \hat{A}^T P & P \hat{B} & \hat{C}^T \\
P \hat{B}^T & -\gamma I & \hat{D}^T \\
\hat{C} & \hat{D} & -\gamma I
\end{bmatrix} < 0 \quad (21)
$$

And

$$
P_0 > 0, P_0 \hat{A}_0 + \hat{A}_0^T P_0 < 0 \quad (22)
$$

$$
K > 0, \hat{A}_0^T K + K \hat{A}_0 < 0 \quad (23)
$$

Where

$$
\hat{A} = \begin{bmatrix}
\hat{A}_0 & \hat{B}_0 \hat{C}_c \\
-B_c \hat{b}_m \hat{C}_0 & \hat{A}_c + \hat{B}_c \hat{A}_m^T
\end{bmatrix} \quad (24)
$$

And the state space representation of the error system is as follows:

$$
\begin{bmatrix}
\hat{A} \hat{B} \\
\hat{C} \hat{D}
\end{bmatrix} = \begin{bmatrix}
A & 0 & B \\
0 & \hat{A} & \hat{B} \\
\hat{C} & 0 & \hat{D}
\end{bmatrix}
$$

Where

$$
\hat{A} = \begin{bmatrix}
A_0 & B_c \hat{C}_c & 0 & 0 \\
0 & \hat{A}_c + B_c \lambda^T & 0 & 0 \\
0 & 0 & -B_c \hat{b}_m \hat{C}_0 & \hat{A}_c + B_c \lambda^T \\
0 & \hat{b}_m \hat{B}_c & 0 & \hat{B}_c
\end{bmatrix} \quad (26)
$$

$$
\hat{B} = \begin{bmatrix}
0 \\
\hat{b}_m \hat{B}_c \\
0 \\
\hat{B}_c
\end{bmatrix} \quad (27)
$$

$$
\hat{C} = [0 \ C_c \ 0 \ \hat{C}_c] \\
\hat{D} = 0
$$

(28) (29)

We can partition the matrix $P$ according to (30) and insert (26)-(29) into (21) to acquire the final matrix inequality.

$$
P = \begin{bmatrix}
P_{11} & P_{12} & P_{13} & P_{14} \\
P_{21} & P_{22} & P_{23} & P_{24} \\
P_{31} & P_{32} & P_{33} & P_{34} \\
P_{41} & P_{42} & P_{43} & P_{44}
\end{bmatrix} > 0 \quad (30)
$$

Note: In reduced system, matrices $\hat{A}_c, \hat{B}_c, \hat{C}_c$ are known based on (12-14) and their dimensions depend on the order of the reduced system. $\hat{b}_m$ can be chosen arbitrary. After solving optimization problem, $(\hat{A}_0, \hat{B}_0, \hat{C}_0, \hat{A})$ are obtained.

If the matrix inequality constraint (22) and (23) are satisfied, then the minimum phase property of the reduced system is guaranteed. In other words, equation (22) causes the real part eigenvalues of $\hat{A}_0$ to be negative. As a result the zeros of $\hat{G}(s)$ are at LHP. Thus, the reduced system $\hat{G}(s)$ is minimum phase and the aim of the paper is satisfied.

The matrix inequality (21)-(23) are not a linear matrix inequality because of the bilinear terms. Thus, LMI methods cannot be applied to find a solution. Suggested solution is to change nonlinear problem into two simpler optimization procedures which are linear in decision variables.

An iterative two-step algorithm can be used as follows:

- Find an initial estimation for $\hat{G}(s)$ from other classical techniques.
- Choose an initial, arbitrary and proper upper bound ($\gamma_{ini}$) for gamma.
- Keep $(\hat{A}_0, \hat{B}_0, \hat{C}_0, \hat{A})$ constant and minimize $\gamma$ with respect to $(P, P_0, K)$ subject to inequalities (21), (22), (23), (30) and (31):

$$
0 < \gamma = \gamma_{opt} < \gamma_{ini} \quad (31)
$$

(The optimum $\gamma_{opt} = \gamma_1$ computed in this step is used as initial $\gamma$ for the next step).

- Keep $(P, P_0, K)$ constant, minimize $\gamma$ with respect to $(\hat{A}_0, \hat{B}_0, \hat{C}_0, \hat{A})$ and subject to inequalities (21), (22), (23) and (31). (The optimum $\gamma_{opt} = \gamma_2$ computed in this step is also used as initial $\gamma$ for step 3)

- Repeat steps 3 and 4 until the difference between $\gamma_1$ and $\gamma_2$ is less than a prescribed tolerance $\epsilon$ (i.e. stop condition is: $|\gamma_1 - \gamma_2| < \epsilon$)

IV. SIMULATION RESULTS

In this section, an example is presented to illustrate the performance of the proposed method. The corresponding algorithm has been solved using the MATLAB LMI toolbox [18].

**Example:** A 4th-order system

Consider the forth-order system;

$$
\hat{G}(s) = \frac{s^3 + 7s^2 + 14s + 8}{s^4 + 10s^3 + 35s^2 + 50s + 25}
$$

In this system, $n = 4$, $m = 3$, $r = 1$. Now, we try to reduce this system to a 3rd-order system where $n' = 3$, $m' = 2$, $r' = 1$. Based on the expressed state-space realization in section 2-B:

$$
Q(s) = s + 3
$$

$$
\frac{R(s)}{N(s)} = \frac{1}{s^3 + 7s^2 + 14s + 8}
$$

Therefore, the following matrices are known:

$$
A_c = 0, B_c = 1, C_c = 1, \lambda = -3, b_m = 1
$$

$$
A_0 = \begin{bmatrix}
-7 & -14 & -8 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
$$

(35)
\[ B_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \]
\[ C_0 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \]
\[ D_0 = 0 \]

In the reduced model, we have:
\[ \hat{A}_c = 0, \hat{B}_c = 1, \hat{C}_c = 1, \hat{B}_m = 1 \]
(36)

The results which are achieved after running the iterative minimization algorithm are as follows:
\[ \hat{A}_0 = \begin{bmatrix} -16.75 & -1.99 \\ -2.83 & -18.18 \end{bmatrix} \]
\[ \hat{B}_0 = \begin{bmatrix} -18.37 \\ 7.76 \end{bmatrix} \]
\[ \hat{C}_0 = \begin{bmatrix} -31.47 & -38.74 \end{bmatrix} \]
\[ \hat{\lambda} = 9.99 \]

Thus, the following can be resulted:
\[ \hat{A} = \begin{bmatrix} -16.75 & -1.99 & -18.37 \\ -2.83 & -18.18 & 7.76 \\ 31.74 & 38.74 & 9.99 \end{bmatrix} \]
(38)

\text{eigenvalue} \ (\hat{A}_0) = -14.98, -19.94
(39)

\text{eigenvalue} \ (\hat{A}) = -13.247, -5.846 \pm 6.192
(40)

As stated before, the eigenvalues of \( \hat{A}_0 \) and \( \hat{A} \) are the zeros and poles of the transfer function \( \hat{G}(s) \), respectively. It is obvious that the minimum phase characteristic of the main system preserved, in both poles and zeroes of the reduced system are at LHP. In Fig.3, frequency response of the main system and the reduced one are plotted. According to the figure, it is obvious that the minimum phase characteristic of the system is preserved and the bode plot of the main system and the estimated one are close to each other.

In this paper, un-weighted model order reduction is applied, while in weighted model reduction, it is possible to match the frequency response of the main system and the reduced one in a desired frequency range which will gain better results on that specified range. In this paper weighted model reduction is ignored to prevent computational complexity. However, it is straightforward to take weights into account. The optimization problem will change to [6]:
\[ \min_{\hat{G}(s)} \| W(s) (G(s) – \hat{G}(s)) V(s) \| \]
(41)

Where \( V(s) \) is the input frequency weight and \( W(s) \) is the output frequency weight.

V. CONCLUSION

Model order reduction is defined as an optimization problem. The matrix inequality approach is used to minimize the \( H_{\infty} \) norm of the difference between the original system and the reduced one.

The main point of this paper was to propose an extra matrix inequality constraint to guaranty that the minimum phase characteristic of a system preserves after reduction. To handle it, a special state-space realization of the system is defined in a way to be able to apply condition on the zeros of the reduced system.

This problem was not a linear matrix inequality because of bilinear terms. As a solution, two steps iterative schemes are used to change nonlinear problem into linear matrix inequality. Then, the method applied to an example and its efficiency is shown.

The main drawback of iterative LMIs is that we cannot guaranty that the solution converges toward the global minimum. Moreover, priori knowledge of the variables is required. Obtaining a non-iterative \( H_{\infty} \)-based model reduction algorithm under the proposed matrix inequality constraint is suggested as the future work of the present paper.

REFERENCE


